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AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZN 103/82 FEBRUARI

S. LIPNISKY

THE HECKE ALGEBRA OF A NEAR HEXAGON

amsterdam 1982

stichting
mathematisch
centrum

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Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 51E30

The Hecke algebra of a near hexagon

by

S. Lipnisky

ABSTRACT

We compute the irreducible representations of the Hecke algebra of a near hexagon H, which is a certain algebra of endomorphisms of the vector space spanned by the maximal flags of H. Some divisibility conditions concerning the parameters of the near hexagon are obtained.

KEY WORDS & PHRASES: *near hexagon, Hecke algebra*

1. INTRODUCTION

A *linear incidence system* is a system (P,L) of points P and a collection L of subsets of P called lines such that every pair of points lie on at most one line. We say that two points p and q are at distance d if their distance in the collinearity graph $\Gamma(P,L)$ is d.

A *near hexagon* (see [4]) is a linear incidence system (P,L) such that

(i) Every point lies on at least one line

(ii) The distance between two points is at most three

(iii) For any point p and line ℓ, there is a unique point on ℓ nearest p.

A near hexagon is said to have *order* (s,t) if each point lies on 1+t lines and each line contains 1+s points, and it is called *regular* with parameters (s,t,t_2) if it has order (s,t) and for each pair of points p and q at distance two, there are 1+t lines through p containing a neighbour of q.

2. DEFINITION AND STRUCTURE OF THE ALGEBRA

Consider a regular near hexagon with parameters (s,t,t_2). Define $F = \{(p,\ell) \in P \times L \mid p \in \ell\}$. Elements of F are called *flags*. Let V be the $\mathbb{C}$-vector space with F as basis and define endomorphisms σ and τ of V by

$$\sigma((p,\ell)) = \sum_{\substack{p'\in\ell \\ p'\neq p}} (p',\ell), \qquad \tau((p,\ell)) = \sum_{\substack{p\in m \\ m\neq\ell}} (p,m).$$

The Hecke algebra H is defined to be the subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$ generated by σ and τ.

<u>LEMMA 1</u>. *The maps σ and τ satisfy the relations*

(1) $$\sigma^2 = s.1+(s-1)\sigma, \qquad \tau^2 = t.1+(t-1)\tau.$$

<u>PROOF</u>. Let $(p,\ell) \in F$, $\ell = \{p,p_1,\ldots,p_s\}$. Then

$$\sigma^2((p,\ell)) = \sum_{i=1}^{s} [(p,\ell) + \sum_{j\neq i} (p_j,\ell)] = s(p,\ell)+(s-1)\sigma((p,\ell))$$

and similar for τ. □

Two flags $x,y \in I$ are said to be adjacent (notation $x \sim y$) if $x \cap y \in L$ or $x \cap y \in P$ (i.e. $x = (p,\ell)$, $y = (q,m)$ and $p = q$ or $\ell = m$). A *gallery* of length k from x to y is a sequence $x = x_0, x_1, \ldots, x_k = y$ of flags such that $x_{i-1} \sim x_i$ for all $i = 1,\ldots,k$. We define $d(x,y)$ as the minimal length of a gallery from x to y.

Define further subsets $S_k(x)$ and $T_k(x)$ of F as follows. A flag y belongs to $S_k(x)$ (resp. $T_k(x)$) iff there exists a minimal gallery from x to y, $x = x_0, x_1, \ldots, x_k = y$, such that $x_{k-1} \cap y \in L$ (resp. $x_{k-1} \cap y \in P$). Thus

$$\sigma x = \sum_{y \in S_1(x)} y, \qquad \tau x = \sum_{y \in T_1(x)} y.$$

<u>THEOREM 1</u>. *Assume that the diameter of H is three. The Hecke algebra has then dimension 12 and the defining relations are* (1) *and*

$$(2) \qquad (\sigma\tau)^3 - (\tau\sigma)^3 + t_2(s-1)((\tau\sigma)^2 - (\sigma\tau)^2) + st_2(\sigma\tau - \tau\sigma) = 0$$

<u>PROOF</u>. Define $\sigma_k, \tau_k \in \mathrm{End}_{\mathbb{C}}(V)$ by

$$\sigma_k(x) = \sum_{y \in S_k(x)} y, \qquad \tau_k(x) = \sum_{y \in T_k(x)} y.$$

We have $S_i \cap S_j = \emptyset$ $(i \neq j)$, $T_i \cap T_j = \emptyset$ $(i \neq j)$, $S_6 = T_6$, $S_i \cap T_j = \emptyset$ $((i,j) \notin \{(4,4), (6,6)\})$. From $\sigma_6 = \tau_6$, $\sigma\tau_5 = \sigma_6$, $\tau\sigma_5 = (t_2+1)\sigma_6 + t_2\sigma_5$ and other relations between the maps σ_i and τ_j, $(i,j) \in \{1,\ldots,6\}$, we get (2). Thus $\dim H \leq 12$.

On the other hand, $\Gamma = \{1, \sigma_1 = \sigma, \ldots, \sigma_5, \tau_1 = \tau, \ldots, \tau_6\}$ is a linearly independent set of endomorphisms in H, so $\dim H = 12$ and Γ is a basis for H. □

3. REPRESENTATIONS OF H

<u>THEOREM 2</u>. *The Hecke algebra* H *has four representations of degree* 1, $\mathrm{ind} = \lambda_1$, $\mathrm{st} = \lambda_2$, λ_3, λ_4, given by

$$\lambda_1(\sigma) = s, \quad \lambda_1(\tau) = t; \quad \lambda_2(\sigma) = -1, \quad \lambda_2(\tau) = -1;$$
$$\lambda_3(\sigma) = s, \quad \lambda_3(\tau) = -1; \quad \lambda_4(\sigma) = -1, \quad \lambda_4(\tau) = t$$

and two irreducible representations X_1 and X_2 of degree 2 given by

$$X_i(\sigma) = S_i := \begin{pmatrix} -1 & a_i \\ 0 & s \end{pmatrix}, \quad X_i(\tau) = T_i := \begin{pmatrix} t & 0 \\ b_i & -1 \end{pmatrix}$$

where a_i and b_i are complex numbers such that
$a_i b_i = s+t+\frac{1}{2}\{t_2(s-1)-\cos(\pi i)[4s(t-t_2)+t_2^2(s-1)^2]^{\frac{1}{2}}\}$.

PROOF. The relations (1) and (2) are satisfied if we replace σ by $\lambda_i(\sigma)$, resp. $X_i(\sigma)$ and τ by $\lambda_i(\tau)$, resp. $X_i(\tau)$. Since dim H = 12, we have found all irreducible representations. In particular, we see that H is semisimple. □

In order to find the values taken by the characters χ_i of X_i $(i = 1,2)$, we follow [2], §3. Put $\alpha_i = a_i b_i - s - t$ and let $\alpha_i = \alpha$, $a_i = a$, $b_i = b$, $S_i = S$, $T_i = T$, $\chi_i = \chi$.

The eigenvalues of ST and TS are then $\frac{1}{2}\{\alpha\pm(\alpha^2-4st)^{\frac{1}{2}}\}$. Let P be a matrix such that

$$D := P^{-1}STP = \begin{pmatrix} \alpha+(\alpha^2-4st)^{\frac{1}{2}} & 0 \\ 0 & \alpha-(\alpha^2-4st)^{\frac{1}{2}} \end{pmatrix}$$

and put $S' = P^{-1}SP$, $T' = P^{-1}TP$. We have $T' = S'^{-1}D$ and if $S' = \begin{pmatrix} x & u \\ v & y \end{pmatrix}$, then from Tr $S' = s-1$, Tr $T' = t-1$, we get

$$x = \frac{2s(t-1)+(s-1)(\alpha+(\alpha^2-4st)^{\frac{1}{2}})}{2(\alpha^2-4st)^{\frac{1}{2}}}, \quad y = s-1-x = \frac{-2s(t-1)+(s-1)(-\alpha+(\alpha^2-4st)^{\frac{1}{2}})}{2(\alpha^2-4st)^{\frac{1}{2}}}.$$

Hence $\chi((\sigma\tau)^k) = \chi((\tau\sigma)^k) = \mathrm{Tr}(D^k)$,

$$\chi((\sigma\tau)^k\sigma) = \mathrm{Tr}((ST)^k S) = \mathrm{Tr}(D^k S'),$$
$$\chi((\tau\sigma)^k\tau) = \mathrm{Tr}(T(ST)^k) = \mathrm{Tr}(T'D^k).$$

COROLLARY. *The character* χ *takes the following values*

$$\chi(1) = 2, \quad \chi(\sigma) = s-1, \ \chi(\tau) = t-1, \quad \chi(\sigma\tau) = \alpha,$$

$$\chi((\sigma\tau)^2) = \alpha^2 - 2st, \quad \chi((\sigma\tau)^3) = \alpha^3 - 3\alpha st,$$

$$\chi(\sigma\tau\sigma) = s(t-1)+\alpha(s-1), \quad \chi((\sigma\tau)^2\sigma) = \alpha s(t-1)+(s-1)(\alpha^2-st),$$

$$\chi(\tau\sigma\tau) = t(s-1)+\alpha(t-1), \quad \chi((\tau\sigma)^2\tau) = \alpha t(s-1)+(t-1)(\alpha^2-st). \quad \square$$

Let ϕ_V be the character of H affended by the representation of H on V. Then $\phi_V = \Sigma n_\psi \psi$ where the sum is over all irreducible characters of H. The multiplicities n_ψ are non-negative integers, $\phi_V(1) = |\mathcal{F}|$ and $\phi_V(\gamma) = 0$ for all $1 \neq \gamma \in \Gamma$ since $\gamma(x)$ $(x \in \mathcal{F})$ is a sum of flags distinct from x. Applying both sides of $\phi_V = \Sigma n_\psi \psi$ to the basis vectors, we get a system of equations involving the integers n_ψ.

Moreover, the element $\Sigma_{x\in\mathcal{F}} x_{\mathcal{F}}$ of V is an eigenvector of σ and τ with respect to the corresponding eigenvalues s, resp. t. The multiplicity is the rank of the corresponding eigenspace, which is one, since the geometry defined by $\mathcal{P}$ and $\mathcal{L}$ is connected. Thus $n_{ind} = 1$.

One can also show ([3], lemma 32) that $n_{st} = |\mathcal{F}| - |\mathcal{P}| - |\mathcal{L}| + 1$.

As a consequence of the equations, we get the following divisibility conditions.

<u>LEMMA 2</u>. *Let H be a regular near hexagon with parameters* (s,t,t_2) and let α_i be defined as above. Then

$$n_{\chi_i} = \frac{(\alpha_j+(\alpha_j-1)\frac{s(t-t_2)}{1+t_2})st(s+1)(t+1)}{(\alpha_i+s+t)(\alpha_j-\alpha_i)} \in \mathbb{Z} > 0 \ (i\neq j, i,j\in\{1,2\}).$$

If $t = s^3 + t_2(s^2-s+1)$ *(the Mathon bound), then* $\alpha_1 = s^2+st_2-t_2$, $\alpha_2 = -s^2$ *and*

$$n_{\chi_1} = \frac{s^2(s+1)(s^2-s+1)^2(s+t_2+1)(s^3+t_2(s^2-s+1))}{(1+t_2)(2s^2+st_2-t_2)}.$$

4. REPRESENTATIONS ON LINES AND QUADS

Fix a point $p \in \mathcal{P}$ and consider the lines through, resp. quads on that point. Denote these sets by L, resp. Q. Then $|L| = t+1$, $|Q| = \frac{t(t+1)}{t_2(t_2+1)}$.

We can define again the set of all flags as

$$F = \{(\ell,q)\in L\times Q \mid \ell\in q\}.$$

We have $|F| = \dfrac{t(t+1)}{t_2}$.

Define $\sigma_L((\ell,q)) = \sum\limits_{\substack{m\in q\\ m\neq\ell}} (m,q)$, $\sigma_Q((\ell,q)) = \sum\limits_{\substack{\ell\in s\\ s\neq q}} (\ell,s)$.

The defining relations of the Hecke algebra are

$$\sigma_L^2 = t_2\cdot 1+(t_2-1)\sigma_L, \qquad \sigma_Q^2 = (\frac{t}{t_2}-1)\cdot 1+(\frac{t}{t_2}-2)\sigma_Q,$$

$$(\sigma_Q\sigma_L)^2 = t_2\sigma_Q\sigma_L\sigma_Q-\sigma_L\sigma_Q\sigma_L+t_2\sigma_L\sigma_Q.$$

In this case, dim H = 7 and the irreducible representations of H are the following

$$\text{ind}:\ 1\mapsto 1,\ \sigma_L\mapsto t_2,\ \sigma_Q\mapsto \frac{t}{t_2}-1$$

$$\text{st}:\ 1\mapsto 1,\ \sigma_L\mapsto -1,\ \sigma_Q\mapsto -1$$

$$\lambda:\ 1\mapsto 1,\ \sigma_L\mapsto t_2,\ \sigma_Q\mapsto -1$$

$$\chi:\ 1\mapsto 2,\ \sigma_L\mapsto \begin{pmatrix} t_2 & x\\ 0 & -1\end{pmatrix},\ \sigma_Q\mapsto \begin{pmatrix} \frac{t}{t_2}-1 & 0\\ y & -1\end{pmatrix}$$

where $x,y\in\mathbb{C}$ and $xy = -(t+1)$.

As before $n_{ind} = 1$, $n_{st} = |F| - |L| - |Q| + 1 = \dfrac{t(t-t_2)}{1+t_2}$.

An easy computation yields the other two multiplicities, which are $n_\chi = t$,

$$n_\lambda = (t+1)(\frac{t}{t_2(t_2+1)}-1).$$

These equations do not give us any new conditions, as was to be expected, since any 2-(v,k,1) design has this Hecke algebra.

One can try to construct the Hecke algebra on points, lines and quads (or on lines and quads). However, no new information is expected, since the vector space becomes much bigger and the defining relations are harder to find.

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